Stochastic Quantization of $(\lambda\varphi^4)_d$ Scalar Theory: Generalized Langevin Equation with Memory Kernel

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Abstract

We review the method of stochastic quantization for a scalar field theory. We first give a brief survey for the case of self-interacting scalar fields, implementing the stochastic perturbation theory up to the one-loop level. The divergences therein are taken care of by employing the usual prescription of the stochastic regularization, introducing a colored random noise in the Einstein relations. We then extend this formalism to the case where we assume a Langevin equation with a memory kernel. We have shown that, if we also maintain the Einstein's relations with a colored noise, there is convergence to a non-regularized theory.

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1   Introduction

In the last century Parisi and Wu introduced the stochastic quantization [1]. This new quantization method differs from the others, the canonical and the path integral field quantization, based in the Hamiltonian and the Lagrangian, respectively, in many aspects. The method starts from a classical equation of motion, but not from Hamiltonian or Lagrangian, and consequently can be used to quantize dynamical systems without canonical formalism. Furthermore, it is useful in situations where the others methods lead to difficult problems and can bring us new important results. As stressed by Rumpf [2], since the stochastic quantization is quite different from the other quantization methods, it can reveal new structural elements of the classical theory which so far have gone unnoticed. The main idea of the stochastic quantization is that a d-dimensional quantum system is equivalent to a $(d+1)$-dimensional classical system with random fluctuations. Some of the most important papers in the subject can be found in Ref. [3]. A brief introduction to the stochastic quantization can be found in the Ref. [4] and Ref. [5]. See also the Ref. [6].

In this paper we would like to discuss the stochastic quantization method for the $(\lambda\varphi^4)_d$ scalar theory using a Langevin equation with a memory kernel and a colored noise. We inquiry which kind of theory appears in the asymptotic limit of this non-Markovian process at one-loop level. Does this stationary, Gaussian, non-Markovian theory reaches the equilibrium with the structure of its ultraviolet divergences under control?

The method of stochastic quantization can be summarized by the following steps. First, starting from a field defined in Minkowski spacetime, after analytic continuation to imaginary time, the Euclidean counterpart, i.e., the field living in an Euclidean space, is obtained. Second, it is introduced a monotonically crescent Markov parameter, called in the literature "fictitious time" and also a random noise field $\eta(\tau, x)$, which simulates the coupling between the classical system and a heat reservoir. It is assumed that the fields defined at the beginning in a d-dimensional Euclidean space also depends on the Markov parameter, therefore the field and a random noise field are defined in a $(d+1)$-dimensional manifold. One starts with the system out of equilibrium at an arbitrary initial state. It is then forced into equilibrium assuming that its evolution is governed by a Markovian Langevin equation with a white random noise field. In fact, this evolution is described by a process which is stationary, Gaussian and Markovian. Finally, the n-point correlation functions of the theory in the $(d+1)$-dimensional space are defined by performing averages over the random noise field with a Gaussian distribution, that is, performing the stochastic averages $\langle\, \varphi(\tau_1, x_1)\varphi(\tau_2, x_2)...\varphi(\tau_n, x_n)\,\rangle_\eta$. The n-point Schwinger functions of the Euclidean d-dimensional theory are obtained evaluating these n-point stochastic averages $\langle\, \varphi(\tau_1, x_1)\varphi(\tau_2, x_2)...\varphi(\tau_n, x_n)\,\rangle_\eta$ when the Markov parameter goes to infinity ($\tau \to \infty$), and the equilibrium is reached. This can be proved in different ways for the particular case of Euclidean scalar field theory. One can

1

use, for instance, the Fokker-Planck equation [7] [8] associated with the equations describing the stochastic dynamic of the system. A diagrammatical technique [9] has also been used to prove such equivalence.

The original method proposed by Parisi and Wu was extended to include process with fermions [10] [11] [12]. The first question that appears in this context is if make sense the Brownian problem with anticommutating numbers. It can be shown that, for massless fields, there will not be a convergence factor after integrating the Markovian Langevin equation. Therefore the equilibrium is not reached. One way of avoiding this problem is to introduce a kernel in the Langevin equation describing the evolution of two Grassmannian fields.

Usually, the Parisi-Wu schemes of quantization applied to bosonic and fermionic fields converge towards non-regularized theories. In order to obtain finite results, the original stochastic process is modified to a non-Markovian process, taking into account a colored noise in the Einstein relations [13] [14]. In few words, the idea of the stochastic regularization is to start from an interacting theory, then construct Langevin tree-graphs, where each leg of which ends in a regularized noise factor. Since it is possible to obtain the loops of the theory by contracting the noise factors, one ends up with a theory where every closed loop contains at least some power of the regulator. With this modification, one can show that the system converges towards a regularized theory. The next step to construct a finite theory is to use for example a minimal-subtraction scheme in which the ultraviolet divergent contributions are eliminated.

However, different types of non-Markovian processes are allowed in principle; for instance, we can modify the Langevin equation introducing a memory kernel. To make sure that the generalized Langevin equation can also be used as a quantization tool, one must check that the process converges in the asymptotic limit and also that converges to the correct equilibrium distribution. In this paper we address the following question: which kind of interacting field theory appears in the asymptotic limit of this non-Markovian process at one-loop level? We have shown that although a system with a stationary, Gaussian, non-Markovian Langevin equation with a memory kernel and a colored noise converges to equilibrium, we obtain a non-regularized theory.

The organization of the paper is the following: in section 2 we briefly discuss the Markovian and the non-Markovian Langevin equation in stochastic processes. We discuss in section 3 the stochastic quantization for the free scalar theory. The stochastic quantization for the $(\lambda \varphi^4)_d$ self-interaction scalar theory with the usual stochastic regularization method is presented in section 4. In section 5 assuming a Langevin equation with a memory kernel, we study the free and the interacting field theory that appears in the asymptotic limit of the non-Markovian process. In section 6, we developed a Fokker-Planck approach in order to understand the results obtained in the previous section. Conclusions are given in section 7. To simplify the calculations we assume the units to be such that $\hbar = c = k_B = 1$.

2 Markovian and non-Markovian Langevin equation

Let us briefly discuss the role of the Langevin equation in non-equilibrium statistical mechanics [15]. One way to treat the dynamics of non-equilibrium systems is to use the theory of Brownian motion. The fundamental equation is the Langevin equation, and it contains both frictional and random forces. The equation of motion for a Brownian particle is given by

$$m\frac{d}{dt}\mathbf{v}(t) = -\xi\,\mathbf{v}(t) + \delta\mathbf{F}(t), \tag{1}$$

where $-\xi\,\mathbf{v}(t)$ is the frictional force and $\delta\,\mathbf{F}(t)$ is the stochastic or fluctuating force, also called the random force. In Eq.(1), m is the mass of the Brownian particle immersed in the fluid. Note that the total force that acts on each Brownian particle has been partioned into a friction part and also a fluctuating part. The effects of the fluctuation force can be summarized by given the Einstein relations, namely,

$$\langle\,\delta\mathbf{F}(t)\,\rangle = 0 \tag{2}$$

and also

$$\langle\,\delta\,F^i(t)\,\delta F^j(t')\,\rangle = 2B\delta_{ij}\delta(t - t'), \tag{3}$$

where B is the measure of the strength of the fluctuating force. The Eq.(2) tell us that the random force is zero on average and the delta function in Eq.(3) indicates that there is no correlation between impacts in any two distinct time intervals. Since the frictional force depends just on the velocity of the particle and not on its earlier values, we are describing a Markovian process. The Langevin equation can be solved to give

$$\mathbf{v}(t) = \exp\left(-\frac{\xi}{m}\,t\right)\mathbf{v}(0) + \frac{1}{m}\int_0^t ds\,\exp\left(-\frac{\xi}{m}(t - s)\right)\delta\mathbf{F}(s). \tag{4}$$

Using the fact that the fluctuating force has a Gaussian distribution defined by its first and seconds moments, it is possible to show that the mean squared velocity of each Brownian particle is

$$\langle\,v^2(t)\,\rangle = \exp\left(-\frac{2\xi}{m}\,t\right)v^2(0) + \frac{B}{\xi\,m}\left(1 - \exp\left(-\frac{2\xi}{m}\right)t\right), \tag{5}$$

where the brackets $\langle...\rangle$ in the Eq. (5) denote stochastic averaging.

There are two points that we would like to stress. First, it is assumed that the fluctuating force $\delta\mathbf{F}(t)$ has a Gaussian distribution. Therefore it is possible to relate the strength B of the random noise to the magnitude ξ of the friction $(B = \xi T)$, where T is the temperature of the bath. This

is the well known fluctuation-dissipation theorem. Note that the Langevin equation considered in Eq.(1) is called Markovian, since the frictional force $(-\xi \mathbf{v}(t))$ at t is proportional to the velocity at t and the noise is delta-function correlated or white. Being more specific, the Fourier transform of the correlation function of the noise, that is, the spectral density, is independent of frequency.

The Langevin equation may be generalized to non-Markovian types assuming that the friction at time t can depend on the history of the velocity $\mathbf{v}(s)$ for times s that are earlier than t [16] [17] [18]. In this situation, the friction coefficient must be replaced by a memory function. The Langevin equation with a dissipative memory kernel, that we will call a generalized Langevin equation, reads

$$m\frac{d}{dt}\mathbf{v}(t) = -\int_0^\infty ds\, M(s)\mathbf{v}(t-s) + \delta\mathbf{F}(t). \tag{6}$$

In the paper we will assume a non-Markovian Langevin equation with a memory kernel and a colored noise. We will examine the interacting scalar field theory that appears in the asymptotic limit of this non-Markovian process. To bring the reader to this new approach let us first consider the standard stochastic quantization of a free scalar field theory.

3   Stochastic quantization for a free scalar theory

Let us consider a neutral scalar field with a $(\lambda\varphi^4)$ self-interaction, defined in a d-dimensional Minkowski spacetime. The vacuum persistence functional is the generating functional of all vacuum expectation value of time-ordered products of the theory. The Euclidean field theory can be obtained by analytic continuation to imaginary time supported by the positive energy condition for the relativistic field theory. In the Euclidean field theory, we have the Euclidean counterpart for the vacuum persistence functional, that is, the generating functional of complete Schwinger functions. Actually, the $(\lambda\varphi^4)_d$ Euclidean theory is defined by these Euclidean Green's functions. The Euclidean generating functional $Z[h]$ is formally defined by the following functional integral:

$$Z[h] = \int [d\varphi]\ \exp\left(-S_0 - S_I + \int d^d x\, h(x)\varphi(x)\right), \tag{7}$$

where the action that usually describes a free scalar field is

$$S_0[\varphi] = \int d^d x\ \left(\frac{1}{2}(\partial\varphi)^2 + \frac{1}{2}m_0^2\,\varphi^2(x)\right), \tag{8}$$

and the interacting part, defined by the non-Gaussian contribution, is

$$S_I[\varphi] = \int d^d x\, \frac{\lambda}{4!}\,\varphi^4(x). \tag{9}$$

4

In Eq.(7), $[d\varphi]$ is a translational invariant measure, formally given by $[d\varphi] = \prod_x d\varphi(x)$. The terms λ and m_0^2 are respectively the bare coupling constant and the squared mass of the model. Finally, $h(x)$ is a smooth function that we introduce to generate the Schwinger functions of the theory by functional derivatives. In the weak-coupling perturbative expansion, which is the conventional procedure, we perform a formal perturbative expansion with respect to the non-Gaussian terms of the action. As a consequence of this formal expansion, all the n-point unrenormalized Schwinger functions are expressed in a powers series of the bare coupling constant λ [19] [20].

The aim of this section is to discuss the stochastic quantization of a free scalar field. It can be shown that it is equivalent to the usual path integral quantization. The starting point of the stochastic quantization to obtain the Euclidean field theory is a Markovian Langevin equation. Assume an Euclidean d-dimensional manifold, where we are choosing periodic boundary conditions for a scalar field and also a random noise. In other words, they are defined in a d-torus $\Omega \equiv T^d$. To implement the stochastic quantization we supplement the scalar field $\varphi(x)$ and the random noise $\eta(x)$ with an extra coordinate τ, the Markov parameter, such that $\varphi(x) \rightarrow \varphi(\tau, x)$ and $\eta(x) \rightarrow \eta(\tau, x)$. Therefore, the fields and the random noise are defined in a domain: $T^d \times R^{(+)}$. Let us consider that this dynamical system is out of equilibrium, being described by the following equation of evolution:

$$\frac{\partial}{\partial \tau} \varphi(\tau, x) = -\frac{\delta S_0}{\delta \varphi(x)}\big|_{\varphi(x)=\varphi(\tau, x)} + \eta(\tau, x), \tag{10}$$

where τ is a Markov parameter, $\eta(\tau, x)$ is a random noise field and S_0 is the usual free action defined in Eq.(8). For a free scalar field, the Langevin equation reads

$$\frac{\partial}{\partial \tau} \varphi(\tau, x) = -(-\Delta + m_0^2)\varphi(\tau, x) + \eta(\tau, x), \tag{11}$$

where Δ is the d-dimensional Laplace operator. The Eq.(11) describes a Ornstein-Uhlenbeck process and we are assuming the Einstein relations, that is:

$$\langle \eta(\tau, x) \rangle_\eta = 0, \tag{12}$$

and for the two-point correlation function associated with the random noise field

$$\langle \eta(\tau, x)\, \eta(\tau', x') \rangle_\eta = 2\delta(\tau - \tau')\, \delta^d(x - x'), \tag{13}$$

where $\langle ... \rangle_\eta$ means stochastic averages. In a generic way, the stochastic average for any functional of φ given by $F[\varphi]$ is defined by

$$\langle F[\varphi] \rangle_\eta = \frac{\int [d\eta] F[\varphi] \exp\left[-\frac{1}{4} \int d^d x \int d\tau\, \eta^2(\tau, x)\right]}{\int [d\eta] \exp\left[-\frac{1}{4} \int d^d x \int d\tau\, \eta^2(\tau, x)\right]}. \tag{14}$$

5

Let us define the retarded Green function for the diffusion problem that we call $G(\tau - \tau', x - x')$. The retarded Green function satisfies $G(\tau - \tau', x - x') = 0$ if $\tau - \tau' < 0$ and also

$$\left[\frac{\partial}{\partial \tau} + (-\Delta_x + m_0^2) \right] G(\tau - \tau', x - x') = \delta^d(x - x')\delta(\tau - \tau'). \tag{15}$$

Using the retarded Green function and the initial condition $\varphi(\tau, x)|_{\tau=0} = 0$, the solution for Eq.(11) reads

$$\varphi(\tau, x) = \int_0^\tau d\tau' \int_\Omega d^d x' \, G(\tau - \tau', x - x')\eta(\tau', x'). \tag{16}$$

In the following we are interested in calculating the quantity $\langle \varphi(\tau, x)\varphi(\tau', x') \rangle_\eta$. Using Eq.(12), Eq.(13) and Eq.(16), we have

$$\langle \varphi(\tau_1, x_1)\varphi(\tau_2, x_2) \rangle_\eta = 2 \int_0^{min(\tau_1, \tau_2)} d\tau' \int_\Omega d^d x' G(\tau_1 - \tau', x_1 - x') \, G(\tau_2 - \tau', x_2 - x'), \tag{17}$$

where $min(\tau_1, \tau_2)$ means the minimum of τ_1 and τ_2. Using a Fourier representation, the two-point correlation function $\langle \varphi(\tau, x)\varphi(\tau', x') \rangle_\eta \equiv D(\tau, x; \tau', x')$ is given by

$$D(\tau, x; \tau', x') = \frac{1}{(2\pi)^d} \int d^d p \, \frac{e^{-ip(x-x')}}{(p^2 + m_0^2)} e^{-(p^2 + m_0^2)(\tau - \tau')}. \tag{18}$$

It is not difficult to show that Eq.(18) can be written as:

$$D(\tau, x; \tau', x') = \frac{1}{(2\pi)^d} \sum_{n=0}^\infty \frac{(-1)^n}{\Gamma(1-n)n!} (\tau - \tau')^n \left(\frac{m_0}{r} \right)^{\frac{d}{2}+n-1} K_{\frac{d}{2}+n-1}(m_0 \, r). \tag{19}$$

where $r = | x - x' |$ and K_ν is the modified Bessel function of order ν.

We can use the Fourier analysis to show that when the Markov parameters τ and τ' go to infinity we recover the standard Euclidean free field theory. Therefore let us define the Fourier transforms for the field and the noise given by $\varphi(\tau, k)$ and $\eta(\tau, k)$. We have respectively

$$\varphi(\tau, k) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int d^d x \, e^{-ikx} \, \varphi(\tau, x), \tag{20}$$

and

$$\eta(\tau, k) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int d^d x \, e^{-ikx} \, \eta(\tau, x). \tag{21}$$

Substituting Eq.(20) in Eq.(8), the free action for the scalar field in the $(d+1)$-dimensional space writing in terms of the Fourier coefficients reads

$$S_0[\varphi(k)]\,|_{\varphi(k)=\varphi(\tau,k)} = \frac{1}{2}\int d^d k\,\varphi(\tau,k)(k^2 + m_0^2)\varphi(\tau,k). \tag{22}$$

Substituting Eq.(20) and Eq.(21) in Eq.(11) we have that each Fourier coefficient satisfies a Langevin equation given by

$$\frac{\partial}{\partial \tau}\varphi(\tau,k) = -(k^2 + m_0^2)\varphi(\tau,k) + \eta(\tau,k). \tag{23}$$

The solution for this equation reads

$$\varphi(\tau,k) = \exp\left(-(k^2 + m_0^2)\tau\right)\varphi(0,k) + \int_0^\tau d\tau' \exp\left(-(k^2 + m_0^2)(\tau - \tau')\right)\eta(\tau',k). \tag{24}$$

Using the Einstein relation, we get that the Fourier coefficients for the random noise satisfies

$$\langle\,\eta(\tau,k)\,\rangle_\eta = 0 \tag{25}$$

and

$$\langle\,\eta(\tau,k)\eta(\tau',k')\,\rangle_\eta = 2(2\pi)^d\delta(\tau - \tau')\,\delta^d(k + k'). \tag{26}$$

Before investigate the interacting field theory, let us calculate the Fourier representation for the two-point correlation function, i.e., $\langle\,\varphi(\tau,k)\varphi(\tau',k')\,\rangle_\eta$. Using Eq.(24), we obtain three contributions to the scalar two-point correlation function. The first one is given by

$$\exp\left(-(k^2 + m_0^2)\,\tau + (k'^2 + m_0^2)\,\tau'\right)\varphi(0,k)\varphi(0,k'), \tag{27}$$

and decay to zero at long time. Let us assume that $\varphi(\tau,k)|_{\tau=0} = 0$. There are also two crossed terms, each first order in the noise Fourier component given by

$$2\,\varphi(0,k)\exp\left(-(k^2 + m_0^2)\tau\right)\int_0^{\tau'} ds\,\exp\left(-(k'^2 + m_0^2)(\tau' - s)\right)\eta(s,k'). \tag{28}$$

Since we are assuming the Einstein relations, i.e., $\langle\,\eta(\tau,x)\,\rangle_\eta = 0$, on averaging on noise, these cross terms vanish. The final term is second-order in the noise Fourier component. Again, the solution subject to the initial condition $\varphi(\tau,k)|_{\tau=0} = 0$ can be used to give

$$\int_0^\tau ds\,\exp\left(-(k^2 + m_0^2)(\tau - s)\right)\eta(s,k)\int_0^{\tau'} d\sigma\,\exp\left(-(k'^2 + m_0^2)(\tau' - \sigma)\right)\eta(\sigma,k'). \tag{29}$$

7

Again averaging on noise and using the Einstein relation given by Eq.(26) we have that this term becomes

$$2\delta^d(k + k') \int_0^{min(\tau, \tau')} ds \exp\left(-(k^2 + m_0)(\tau + \tau' - 2s)\right). \tag{30}$$

Assuming that $\tau = \tau'$ and using $\langle\, \varphi(\tau, k)\varphi(\tau', k')\,\rangle_\eta|_{\tau=\tau'} \equiv D(k, k'; \tau, \tau')$ we have

$$D(k; \tau, \tau) = (2\pi)^d \delta^d(k + k') \frac{1}{(k^2 + m_0^2)} \left(1 - \exp\left(-2\tau(k^2 + m_0^2)\right)\right). \tag{31}$$

In the following, we are redefining the two-point correlation function as $D(k; \tau, \tau) \rightarrow (2\pi)^d D(k; \tau, \tau)$. In the limit when $\tau \rightarrow \infty$ we recover the standard two-point function of the Euclidean free field theory. Before going to the next section, we would like to mention the existence of more general Markovian Langevin equations. We can introduce a kernel defined in the d-torus. The kerneled Langevin equation reads:

$$\frac{\partial}{\partial \tau}\varphi(\tau, x) = -\int d^d y\, K(x, y) \frac{\delta\, S_0}{\delta\, \varphi(y)}|_{\varphi(y)=\varphi(\tau, y)} + \eta(\tau, x). \tag{32}$$

The second moment of the noise field will be modified to:

$$\langle\, \eta(\tau, x)\, \eta(\tau', x')\,\rangle_\eta = 2\delta(\tau - \tau')\, K(x, x'). \tag{33}$$

Choosing an appropriate kernel, it can be shown that all the above conclusions remain unchanged.

4 Stochastic quantization for the $(\lambda\varphi^4)_d$ scalar theory

In this section we will analyze the stochastic quantization for the $(\lambda\varphi^4)_d$ self-interaction scalar theory. In this case the Langevin equation reads

$$\frac{\partial}{\partial \tau}\varphi(\tau, x) = -(-\Delta + m_0^2)\varphi(\tau, x) - \frac{\lambda}{3!}\varphi^3(\tau, x) + \eta(\tau, x). \tag{34}$$

The two-point correlation function associated with the random field is given by the Einstein relation, while the other connected correlation functions vanish, i.e.,

$$\langle\, \eta(\tau_1, x_1)\eta(\tau_2, x_2)...\eta(\tau_{2k-1}, x_{2k-1})\,\rangle_\eta = 0, \tag{35}$$

and also

$$\langle\eta(\tau_1, x_1)...\eta(\tau_{2k}, x_{2k})\,\rangle_\eta = \sum \langle\eta(\tau_1, x_1)\eta(\tau_2, x_2)\,\rangle_\eta \langle\, \eta(\tau_k, x_k)\eta(\tau_l, x_l)\,\rangle_\eta..., \tag{36}$$

where the sum is to be taken over all the different ways in which the $2k$ labels can be divided into k parts, i.e., into k pairs. Performing Gaussian averages over the white random noise, it is possible to prove that

$$\lim_{\tau \to \infty} \langle \varphi(\tau_1, x_1)\varphi(\tau_2, x_2)...\varphi(\tau_n, x_n) \rangle_\eta = \frac{\int [d\varphi]\varphi(x_1)\varphi(x_2)...\varphi(x_n)\, e^{-S(\varphi)}}{\int [d\varphi]\, e^{-S(\varphi)}}, \tag{37}$$

where $S(\varphi) = S_0(\varphi) + S_I(\varphi)$ is the d-dimensional action. This result leads us to consider the Euclidean path integral measure a stationary distribution of a stochastic process. Note that the solution of the Langevin equation needs a given initial condition. As for example

$$\varphi(\tau, x)|_{\tau=0} = \varphi_0(x). \tag{38}$$

Let us use the Langevin equation to perturbatively solve the interacting field theory. One way to handle the Eq.(34) is with the method of Green's functions. We defined the retarded Green function for the diffusion problem in the Eq.(15). Let us assume that the coupling constant is a small quantity. Therefore to solve the Langevin equation in the case of a interacting theory we use a perturbative series in λ. Therefore we can write

$$\varphi(\tau, x) = \varphi^{(0)}(\tau, x) + \lambda\varphi^{(1)}(\tau, x) + \lambda^2\varphi^{(2)}(\tau, x) + ... \tag{39}$$

Substituting the Eq.(39) in the Eq.(34), and if we equate terms of equal power in λ, the resulting equations are

$$\left[\frac{\partial}{\partial \tau} + (-\Delta_x + m_0^2)\right]\varphi^{(0)}(\tau, x) = \eta(\tau, x), \tag{40}$$

$$\left[\frac{\partial}{\partial \tau} + (-\Delta_x + m_0^2)\right]\varphi^{(1)}(\tau, x) = -\frac{1}{3!}\left(\varphi^{(0)}(\tau, x)\right)^3, \tag{41}$$

$$\left[\frac{\partial}{\partial \tau} + (-\Delta_x + m_0^2)\right]\varphi^{(2)}(\tau, x) = -\frac{1}{2!}\left(\varphi^{(0)}(\tau, x)\right)^2 \varphi^{(1)}(\tau, x), \tag{42}$$

and so on. Using the retarded Green function and assuming that $\varphi^{(q)}(\tau, x)|_{\tau=0} = 0$, $\forall q$, the solution to the first equation given by Eq.(40) can be written formally as

$$\varphi^{(0)}(\tau, x) = \int_0^\tau d\tau' \int_\Omega d^dx'\, G(\tau - \tau', x - x')\eta(\tau', x'). \tag{43}$$

The second equation given by Eq.(41) can also be solved using the above result. We obtain

$$\varphi^{(1)}(\tau, x) = -\frac{1}{3!}\int_0^\tau d\tau_1 \int_\Omega d^dx_1\, G(\tau - \tau_1, x - x_1)$$
$$\left(\int_0^{\tau_1} d\tau' \int_\Omega d^dx'\, G(\tau_1 - \tau', x_1 - x')\eta(\tau', x')\right)^3. \tag{44}$$

9

We have seen that we can generate all the tree diagrams with the noise field contributions. We can also consider the n-point correlation function $\langle\,\varphi(\tau_1,x_1)\varphi(\tau_2,x_2)...\varphi(\tau_n,x_n)\,\rangle_\eta$. Substituting the above results in the n-point correlation function, and taking the random averages over the white noise field using the Wick-decomposition property defined by Eq.(36) we generate the stochastic diagrams. Each of these stochastic diagrams has the form of a Feynman diagram, apart from the fact that we have to take into account that we are joining together two white random noise fields many times.

As simple examples let us show how to derive the two-point function in the zeroth order $\langle\,\varphi(\tau_1,x_1)\varphi(\tau_2,x_2)\,\rangle_\eta^{(0)}$, and also the first order correction to the scalar two-point-function given by $\langle\,\varphi(\tau_1,x_1)\varphi(\tau_2,x_2)\,\rangle_\eta^{(1)}$. Using the Eq.(16) and the Einstein relations we have

$$\langle\,\varphi(\tau_1,x_1)\varphi(\tau_2,x_2)\,\rangle_\eta^{(0)} = 2\int_0^{min(\tau_1,\tau_2)} d\tau'\int_\Omega d^dx'\, G(\tau_1-\tau',x_1-x')\,G(\tau_2-\tau',x_2-x'), \qquad (45)$$

which is just Eq.(17). For the first order correction we get:

$$\langle\,\varphi(X_1)\varphi(X_2)\,\rangle_\eta^{(1)} = -\frac{\lambda}{3!}\langle\int dX_3\int dX_4\Big(G(X_1-X_4)G(X_2-X_3) +$$

$$G(X_1-X_3)G(X_2-X_4)\Big)\eta(X_3)\Big(\int dX_5\, G(X_4-X_5)\eta(X_5)\Big)^3\,\rangle_\eta. \qquad (46)$$

where, for simplicity, we have introduced a compact notation:

$$\int_0^\tau d\tau\int_\Omega d^dx \equiv \int dX, \qquad (47)$$

and also $\varphi(\tau,x)\equiv\varphi(X)$ and finally $\eta(\tau,x)\equiv\eta(X)$.

The process can be repeated and therefore the stochastic quantization can be used as an alternative approach to describe scalar quantum fields. We stress here that the stochastic quantization is based in the fact that although one starts with the system out of equilibrium, the Markovian Langevin equation forces it into equilibrium. Moreover, when the thermodynamic equilibrium is reached, the stochastic expectation values will coincide with the Schwinger functions of the Euclidean field theory.
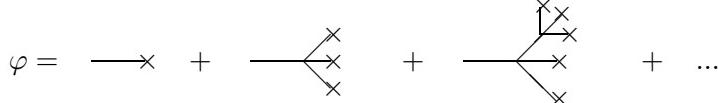
Figure 1: Perturbative expansion for the scalar field where crosses denote noise fields.
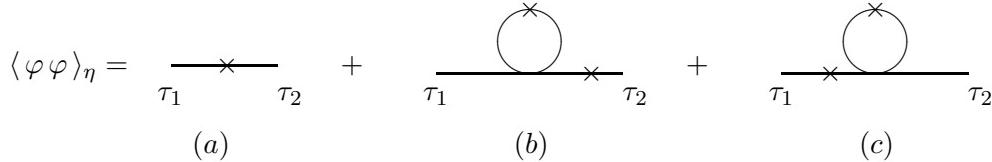

Figure 2: The corrections up to one-loop to the two-point correlation function.

We can represent Eq.(39) graphically as figure (1) (the random noise field is represented by a cross). Using this diagrammatical expansion, it is possible to show that the two-point correlation function up to one-loop level is given by figure (2), where we represent the retarded Green function by a line and the free two-point function by a crossed line. The rules to obtain the algebraic values of the stochastic diagrams are similar to the usual Feynman rules. For instance the two-point function at one-loop level is given by

$$(b) = -\frac{\lambda}{2} \delta^d(k_1 + k_2) \int d^d k \int_0^{\tau_1} d\tau \, G(k_1; \tau_1 - \tau) D(k; \tau, \tau) D(k_2; \tau_2, \tau). \tag{48}$$

$$(c) = -\frac{\lambda}{2} \delta^d(k_1 + k_2) \int d^d k \int_0^{\tau_2} d\tau \, G(k_2; \tau_2 - \tau) D(k; \tau, \tau) D(k_1; \tau_1, \tau). \tag{49}$$

A simple computation shows that we recover the correct equilibrium result at equal asymptotic Markov parameters ($\tau_1 = \tau_2 \to \infty$):

$$(b)|_{\tau_1 = \tau_2 \to \infty} = -\frac{\lambda}{2} \delta^d(k_1 + k_2) \frac{1}{(k_2^2 + m_0^2)} \frac{1}{(k_1^2 + k_2^2 + 2m_0^2)} \int d^d k \frac{1}{(k^2 + m_0^2)}. \tag{50}$$

11

Obtaining the Schwinger functions in the asymptotic limit does not guarantee that we gain a finite physical theory. The next step is to implement a suitable regularization scheme. A crucial point to find a satisfactory regularization scheme is to use one that preserves the symmetries of the original model. In the stochastic regularization method the symmetries of the physical theory is maintained. There are in general two different ways to implement the stochastic regularization. The first one is to start from a Langevin equation with a memory kernel. It is known from the literature [21] that this method can at best only remove two degrees of divergence. Another possibility is to smear only the noise field in the probability functional [22] [23]:

$$\langle F[\varphi] \rangle_\eta = \frac{\int [d\eta] F[\varphi] \exp\left[-\frac{1}{4}\int d^d x \int d\tau \int d\tau'\, \eta(\tau,x) K_\Lambda^{-1}\eta(\tau',x)\right]}{\int [d\eta] \exp\left[-\frac{1}{4}\int d^d x \int d\tau \int d\tau'\, \eta(\tau,x) K_\Lambda^{-1}\eta(\tau',x)\right]}, \tag{51}$$

where K_Λ is a memory kernel. In this case we change the Einstein relations of the noise field to:

$$\langle\, \eta(\tau,x)\,\eta(\tau',x')\,\rangle_\eta = 2\,K_\Lambda(\tau,\tau')\,\delta^d(x-x'). \tag{52}$$

The smearing function should be chosen such that, when $\Lambda \to \infty$:

$$\lim_{\Lambda\to\infty} K_\Lambda(\tau-\tau') = \delta(\tau-\tau'), \tag{53}$$

recovering the usual theory.

Since the Langevin equation is unaffected by the stochastic regularization, the physical field is the same as in the regularized case. However, the zeroth-order two-point correlation function is given by:

$$D(k;\tau,\tau') =$$
$$2\delta^d(k+k')\int_0^\tau ds \int_0^{\tau'} ds'\, G(k;\tau-s)\,G(k;\tau'-s')K_\Lambda(s-s') =$$
$$2\delta^d(k+k')\int_0^\tau ds \int_0^{\tau'} ds' \exp\left(-(\tau+\tau'-s-s')(k^2+m_0^2)\right)K_\Lambda(s-s'). \tag{54}$$

It is possible to prove that a necessary condition that the regularization function K_Λ should satisfy in order to render the divergent loops finite is $K_\Lambda(\tau)\mid_{\tau=0}= 0$. The following series of kernels obeying this condition were proposed:

$$K_\Lambda^{(n)}(\tau) = \frac{1}{2n!}\Lambda^2(\Lambda^2\mid\tau\mid)^n \exp(-\Lambda^2\mid\tau\mid). \tag{55}$$

12

For the case $n = 0$ we obtain, for the free two-point correlation function:

$$\lim_{\tau \to \infty} D(k; \tau, \tau) = \frac{\delta^d(k + k')}{(k^2 + m_0^2)} \frac{\Lambda^2}{(\Lambda^2 + k^2 + m_0^2)}. \tag{56}$$

Since the stochastic diagrams contains crossed lines in its loops, we have that the ultraviolet divergences can be regularized choosing an appropriate n. Note that it is possible to use a different regulator of the type $K_\sigma(\tau) = \frac{1}{2} \sigma \tau^{\sigma-1}$. This regulator scheme is quite similar to the analytic regularization of Bollini et al [24] and Speer [25]. The relation between the cutoff regularization and the analytic regularization procedure has been clarified by Kay [26] and also Svaiter and Svaiter [27] [28] [29] in a series of papers studying the Casimir effect. In the next section we will study a generalized Langevin equation with memory kernel and colored noise.

5   Generalized Langevin equation with colored noise.

It is well known that the Langevin equation given by Eq.(32) is only one particular choice in a large class of relaxation equations. To make sure that the generalized Langevin equation can also be used as a quantization tool, one must check that the process converges in the asymptotic limit and also that converges to the correct equilibrium distribution. Therefore, the aim of this section is to discuss the $(\lambda \varphi^4)_d$ scalar field theory that appears if we start from a Langevin equation with a memory kernel and also a colored noise. To proceed, let us introduce a Langevin equation with a memory kernel given by

$$\frac{\partial}{\partial \tau} \varphi(\tau, x) = - \int_0^\tau ds \, K_\Lambda(\tau - s) \frac{\delta S}{\delta \varphi(x)} |_{\varphi(x) = \varphi(s, x)} + \eta(\tau, x), \tag{57}$$

where the stochastic random field $\eta(\tau, x)$ satisfies the modified Einstein relations.

$$\langle \, \eta(\tau, x) \, \eta(\tau', x') \, \rangle_\eta = 2 K_\Lambda(|\tau - \tau'|) \, \delta^d(x - x'). \tag{58}$$

In this case where $K_\Lambda(|\tau - \tau'|)$ has a width in the fictitious time, the description is Gaussian in spite of being non-Markovian. For the free scalar field we have that the generalized Langevin equation reads

$$\frac{\partial}{\partial \tau} \varphi(\tau, x) = - \int_0^\tau ds \, K_\Lambda(\tau - s)(-\Delta + m_0^2) \varphi(s, x) + \eta(\tau, x). \tag{59}$$

13

Using a Fourier representation for the scalar field and the random noise field we get

$$\frac{\partial}{\partial \tau}\varphi(\tau, k) = -(k^2 + m_0^2)\int_0^\tau ds\, K_\Lambda(\tau - s)\varphi(s, k) + \eta(\tau, k). \tag{60}$$

Following Ref. [16], we define the Laplace transform of the memory kernel:

$$K(z) = \int_0^\infty d\tau\, K_\Lambda(\tau)\, e^{-z\tau}. \tag{61}$$

With the initial condition $\varphi(\tau, k)|_{\tau=0} = 0$, the solution of the Eq.(60) reads:

$$\varphi(\tau, k) = \int_0^\infty d\tau'\, G(k, \tau - \tau')\, \eta(\tau', k), \tag{62}$$

where using the step function $\theta(\tau)$, the Green function $G(k, \tau)$ is defined by:

$$G(k, \tau) \equiv \Omega(k, \tau)\, \theta(\tau). \tag{63}$$

The $\Omega(k, \tau)$ function that appears in Eq.(63) is defined through its Laplace transform:

$$\Omega(k, \tau) = \left(z + (k^2 + m_0^2)K(z)\right)^{-1}. \tag{64}$$

It is easy to see that, in the limit $K_\Lambda(\tau) \to \delta(\tau)$, we obtain the usual result for the Green function. From Eq.(62) and the modified Einstein relations, we get that the free scalar correlation function $D(k; \tau, \tau')$ is given by:

$$
\begin{aligned}
D(k; \tau, \tau') &= \\
&= 2\delta^d(k + k')\int_0^\infty ds \int_0^\infty ds'\, G(k, \tau - s)\, G(k, \tau' - s')\, K_\Lambda(|\, s - s'\,|) \\
&= 2\delta^d(k + k')\int_0^\tau ds \int_0^{\tau'} ds'\, \Omega(k, \tau - s)\, \Omega(k, \tau' - s')\, K_\Lambda(|\, s - s'\,|).
\end{aligned} \tag{65}
$$

To proceed we have to write $D(k; \tau, \tau')$ in a simplified way. Note that the double Laplace transform of the right hand side is given by:

$$
\begin{aligned}
\int_0^\infty d\tau\, e^{-z\tau} \int_0^\infty d\tau'\, e^{-z\tau'} \int_0^\tau ds \int_0^{\tau'} ds'\, \Omega(k, \tau - s)\, \Omega(k, \tau' - s')\, K_\Lambda(|\, s - s'\,|) &= \\
= \Omega(k, z)\, \Omega(k, z')\int_0^\infty ds \int_0^\infty ds'\, e^{-z's'} e^{-zs}\, K_\Lambda(|\, s - s'\,|).
\end{aligned} \tag{66}
$$

Now, with simple manipulations, we get:

$$\int_0^\infty ds \int_0^\infty ds'\, e^{-z's'} e^{-zs} K_\Lambda(|\, s - s'\, |) = \frac{K(z) + K(z')}{z + z'}. \tag{67}$$

Therefore, we get the identity:

$$\int_0^\infty d\tau\, e^{-z\tau} \int_0^\infty d\tau'\, e^{-z'\tau'} \int_0^\tau ds \int_0^{\tau'} ds'\, \Omega(k, \tau - s)\, \Omega(k, \tau' - s')\, K_\Lambda(|\, s - s'\, |) =$$
$$= \Omega(k, z)\, \Omega(k, z') \left(\frac{K(z) + K(z')}{z + z'} \right). \tag{68}$$

Remembering Eq.(64), we can show that:

$$\Omega(k, z)\, \Omega(k, z') \left(\frac{K(z) + K(z')}{z + z'} \right) = \frac{1}{(k^2 + m_0^2)} \left(\frac{\Omega(k, z) + \Omega(k, z')}{z + z'} - \Omega(k, z)\, \Omega(k, z') \right). \tag{69}$$

So, in parallel with result Eq.(67), we finally obtain a very simple expression for $D(k; \tau, \tau')$ in terms of $\Omega(k, \tau)$. We have

$$D(k; \tau, \tau') = 2 \frac{\delta^d(k + k')}{(k^2 + m_0^2)} \left(\Omega(k, |\, \tau - \tau'\, |) - \Omega(k, \tau)\, \Omega(k, \tau') \right). \tag{70}$$

Now, we need an expression for our memory kernel in order to investigate the convergence of Eq.(70). From Eq.(55), we will have, for $n = 0$:

$$K_\Lambda(\tau) = \frac{1}{2} \Lambda^2 \exp(-\Lambda^2 |\, \tau\, |). \tag{71}$$

Then, from Eq.(61), Eq.(64) and Eq.(71), and applying the inverse Laplace transform, we will obtain the following expression for the Ω-function:

$$\Omega(k, \tau) = \left(\frac{\Lambda^2}{\beta} \sin\left(\frac{\beta\tau}{2} \right) + \cos\left(\frac{\beta\tau}{2} \right) \right) \exp\left(-\tau \frac{\Lambda^2}{2} \right). \tag{72}$$

where we have defined a real quantity β given by:

$$\beta \equiv \Lambda \sqrt{2(k^2 + m_0^2) - \Lambda^2}. \tag{73}$$

Similarly, we will have, for the Green function:

$$G(k, \tau) = \left(\frac{\Lambda^2}{\beta} \sin\left(\frac{\beta\tau}{2} \right) + \cos\left(\frac{\beta\tau}{2} \right) \right) \exp\left(-\tau \frac{\Lambda^2}{2} \right) \theta(\tau). \tag{74}$$

15

From the results above, it is easy to see that the free two-point function will be given by:

$$
\begin{aligned}
D(k;\tau,\tau') &= \\
&= 2\,\frac{\delta^d(k+k')}{(k^2+m_0^2)}\left[\left(\frac{\Lambda^2}{\beta}\,\sin\left(\frac{\beta(\tau-\tau')}{2}\right)+\cos\left(\frac{\beta(\tau-\tau')}{2}\right)\right)\exp\left(-\frac{\Lambda^2}{2}\,|\,(\tau-\tau')\,|\right)\right.\\
&\left.-\left(\frac{\Lambda^2}{\beta}\,\sin\left(\frac{\beta\tau}{2}\right)+\cos\left(\frac{\beta\tau}{2}\right)\right)\left(\frac{\Lambda^2}{\beta}\,\sin\left(\frac{\beta\tau'}{2}\right)+\cos\left(\frac{\beta\tau'}{2}\right)\right)\exp\left(-\frac{\Lambda^2}{2}(\tau+\tau')\right)\right]. \quad (75)
\end{aligned}
$$

For $\tau = \tau'$, we get:

$$
D(k;\tau,\tau) = 2\,\frac{\delta^d(k+k')}{(k^2+m_0^2)}\left(1-\left(\frac{\Lambda^2}{\beta}\,\sin\left(\frac{\beta\tau}{2}\right)+\cos\left(\frac{\beta\tau}{2}\right)\right)^2\exp(-\Lambda^2\,\tau)\right). \quad (76)
$$

So, in the limit $\tau \to \infty$, we obtain the following result:

$$
D(k;\tau,\tau) = 2\,\frac{\delta^d(k+k')}{(k^2+m_0^2)}. \quad (77)
$$

which, as we see, does not present an improved ultraviolet behavior. In fact, this equation is much similar to the usual equilibrium result, up to a constant. This is a first evidence that the ultraviolet divergences appearing in the perturbative series when we consider the self-interacting theory may not be regularized. Now, let us study the self-interaction $(\lambda\varphi^4)_d$ scalar field theory within this non-Markovian approach. Now, the Langevin equation reads:

$$
\frac{\partial}{\partial\tau}\varphi(\tau,x) = -\int_0^\tau ds\, K_\Lambda(\tau-s)\left((-\Delta+m_0^2)\varphi(s,x)+\frac{\lambda}{3!}\varphi^3(s,x)\right)+\eta(\tau,x). \quad (78)
$$

We can solve this equation by iteration as before. Then, after equating terms with equal powers in λ, we get:

$$
\left[\frac{\partial}{\partial\tau}+(-\Delta_x+m_0^2)\right]\varphi^{(0)}(\tau,x) = \eta(\tau,x), \quad (79)
$$

$$
\left[\frac{\partial}{\partial\tau}+(-\Delta_x+m_0^2)\right]\varphi^{(1)}(\tau,x) = -\frac{1}{3!}\int_0^\tau ds\, K_\Lambda(\tau-s)\left(\varphi^{(0)}(s,x)\right)^3, \quad (80)
$$

$$
\left[\frac{\partial}{\partial\tau}+(-\Delta_x+m_0^2)\right]\varphi^{(2)}(\tau,x) = -\frac{1}{2!}\int_0^\tau ds\, K_\Lambda(\tau-s)\left(\varphi^{(0)}(s,x)\right)^2\varphi^{(1)}(s,x), \quad (81)
$$

and so on. The solutions of the first two equations can be written as:

$$
\varphi^{(0)}(\tau,x) = \int_0^\tau d\tau'\int_\Omega d^dx'\, G(\tau-\tau',x-x')\eta(\tau',x'). \quad (82)
$$

and

$$
\varphi^{(1)}(\tau, x) = -\frac{1}{3!} \int_0^\tau d\tau_1 \int_\Omega d^d x_1 \, G(\tau - \tau_1, x - x_1) \int_0^{\tau_1} ds \, K_\Lambda(\tau_1 - s)
$$
$$
\left(\int_0^{\tau_1} d\tau' \int_\Omega d^d x' \, G(\tau_1 - \tau', x_1 - x') \eta(\tau', x') \right)^3, \tag{83}
$$

with the Green function given by Eq.(74).

For the n-point correlation functions, the perturbation theory will be similar to the Markovian case, except that for each vertex in the stochastic diagram there is a memory kernel associated. For instance, the one-loop correction for the two-point function is given by $((b)$ and (c) given by figure (2)):

$$
(b) = -\frac{\lambda}{2} \delta^d(k_1 + k_2) \int d^d k \int_0^{\tau_1} d\tau \int_0^\tau ds \, G(k_1, \tau_1 - \tau) D(k; s, s) D(k_2; \tau_2, s) K_\Lambda(\mid \tau - s \mid). \tag{84}
$$

$$
(c) = -\frac{\lambda}{2} \delta^d(k_1 + k_2) \int d^d k \int_0^{\tau_2} d\tau \int_0^\tau ds \, G(k_2, \tau_2 - \tau) D(k; s, s) D(k_1; \tau_1, s) K_\Lambda(\mid \tau - s \mid). \tag{85}
$$

From Eq.(74) and Eq.(75) we may split Eq.(84) in four parts as:

$$
(b) = -\frac{\lambda}{2} \delta^d(k_1 + k_2) \int d^d k \int_0^{\tau_1} d\tau \int_0^\tau ds \left(I_1 + I_2 + I_3 + I_4 \right), \tag{86}
$$

where:

$$
I_1 \equiv \int_0^{\tau_1} d\tau \int_0^\tau ds \, \Omega(k_1, \tau_1 - \tau) \, \Omega(k_2, \tau_2 - s) \, \Omega(k, 0) \, K_\Lambda(\mid \tau - s \mid), \tag{87}
$$

$$
I_2 \equiv \int_0^{\tau_1} d\tau \int_0^\tau ds \, \Omega(k_1, \tau_1 - \tau) \, \Omega(k_2, \tau_2) \, \Omega(k_2, s) \, \Omega(k, 0) \, K_\Lambda(\mid \tau - s \mid), \tag{88}
$$

$$
I_3 \equiv \int_0^{\tau_1} d\tau \int_0^\tau ds \, \Omega(k_1, \tau_1 - \tau) \, \Omega(k_2, \tau_2 - s) \, \Omega^2(k, s) \, K_\Lambda(\mid \tau - s \mid), \tag{89}
$$

$$
I_4 \equiv \int_0^{\tau_1} d\tau \int_0^\tau ds \, \Omega(k_1, \tau_1 - \tau) \, \Omega(k_2, \tau_2) \, \Omega(k_2, s) \, \Omega^2(k, s) \, K_\Lambda(\mid \tau - s \mid). \tag{90}
$$

Using simple calculations, it is easy to show that the terms I_2, I_3 and I_4 have a decaying oscillatory regime for $\tau_1 = \tau_2 \to \infty$. So, we only have to consider the first term. It reads:

$$
I_1 =
$$
$$
= \frac{\Lambda^2}{2} \exp\left(-\frac{\Lambda^2}{2}(\tau_1 + \tau_2)\right) \int_0^{\tau_1} d\tau \left(\frac{\Lambda^2}{\beta_1} \sin\left(\frac{\beta_1(\tau_1 - \tau)}{2}\right) + \cos\left(\frac{\beta_1(\tau_1 - \tau)}{2}\right)\right)
$$
$$
\exp\left(-\frac{\Lambda^2}{2}\tau\right) \int_0^\tau ds \left(\frac{\Lambda^2}{\beta_2} \sin\left(\frac{\beta_2(\tau_2 - s)}{2}\right) + \cos\left(\frac{\beta_2(\tau_2 - s)}{2}\right)\right) \exp\left(\frac{3\Lambda^2}{2}s\right). \tag{91}
$$

After simple manipulations [30] and a tedious algebra, we obtain the final result, in the limit $\tau_1 = \tau_2 \to \infty$:

$$(b)|_{\tau_1=\tau_2\to\infty} = -2\lambda \frac{\delta^d(k_1 + k_2)}{(k_2^2 + m_0^2)} f(\Lambda; \beta_1, \beta_2) \int d^d k \frac{1}{(k^2 + m_0^2)}, \tag{92}$$

where:

$$f(\Lambda; \beta_1, \beta_2) \equiv \left(\frac{(72\,\Lambda^4 + 9\,\beta_1^2 - \beta_2^2)\,\Lambda^4 + \beta_2^2(\beta_1^2 - \beta_2^2)}{(4\Lambda^4 + (\beta_1 - \beta_2)^2)(4\Lambda^4 + (\beta_1 - \beta_2)^2)} \right) \left(\frac{2\,\Lambda^2}{9\Lambda^4 + \beta_2^2} \right), \tag{93}$$

and $\beta_i = \Lambda\sqrt{2(k_i^2 + m_0^2) - \Lambda^2}; i = 1, 2$. So, we see that, although we obtain convergence in the asymptotic limit of the fictitious parameter τ, we do not get a regularized theory. In fact, up to polynomials of Λ, the result is much similar to the usual equilibrium result, Eq.(50). Since the usual stochastic regularization requires the smearing of only the η probability functional, leaving the Langevin equation alone, a natural question that arises is why a theory with a colored, internal noise does not lead to finite results in perturbation theory. Let us try to answer this question within a Fokker-Planck analysis.

6 The Fokker-Planck approach

Although in this paper we study the stochastic perturbation theory using the Langevin equation approach, to understand our results it is more suitable to work within the Fokker-Planck formulation. As we know, correlation functions are introduced as averages over η:

$$\langle\, \varphi(\tau_1, x_1)\varphi(\tau_2, x_2) \cdots \varphi(\tau_n, x_n)\, \rangle_\eta =$$
$$\mathcal{N} \int [d\eta]\varphi(\tau_1, x_1)\varphi(\tau_2, x_2) \cdots \varphi(\tau_n, x_n) \exp\left(-\frac{1}{4} \int d^d x \int d\tau\, \eta^2(\tau, x)\right), \tag{94}$$

where $\mathcal{N}$ is given by

$$\mathcal{N}^{-1} = \int [d\eta] \exp\left(-\frac{1}{4} \int d^d x \int d\tau\, \eta^2(\tau, x)\right). \tag{95}$$

An alternative way to write this average is to introduce the probability density $P[\varphi, \tau]$, which is defined as [8]:

$$P[\varphi, \tau] \equiv \int [d\eta] \exp\left(-\frac{1}{4} \int d^d x \int d\tau'\, \eta^2(\tau', x)\right) \prod_y \delta(\varphi(y) - \varphi(\tau, y)). \tag{96}$$

where, for simplicity, we have absorbed the factor $\mathcal{N}$ in the original functional measure. In terms of $P[\varphi, \tau]$, the equal time correlation functions will read:

$$\langle\, \varphi(\tau_1, x_1)\varphi(\tau_2, x_2) \cdots \varphi(\tau_n, x_n)\, \rangle_\eta = \int [d\varphi]\, \varphi(\tau_1, x_1)\varphi(\tau_2, x_2) \cdots \varphi(\tau_n, x_n) P[\varphi, \tau]. \tag{97}$$

The probability density $P[\varphi, \tau]$ satisfies the Fokker-Planck equation [31]:

$$\frac{\partial}{\partial \tau} P[\varphi, \tau] = \int d^d x \frac{\delta}{\delta \varphi(x)} \left(\frac{\delta}{\delta \varphi(x)} + \frac{\delta S_0}{\delta \varphi(x)} \right) P[\varphi, \tau], \tag{98}$$

with the initial condition:

$$P[\varphi, 0] = \prod_y \delta(\varphi(y)). \tag{99}$$

The stochastic quantization says that we shall have:

$$w. \lim_{\tau \to \infty} P[\varphi, \tau] = \frac{\exp(-S[\varphi])}{\int [d\varphi] \exp(-S[\varphi])}, \tag{100}$$

where the limit is supposed to be taken "weakly" in the sense of the reference [8]. Again following the former reference, for the perturbation theory, we may split the action in two parts:

$$S = S_0 + \lambda S_I. \tag{101}$$

The Green functions of the theory are computed as power series in λ. Similarly, we may expand the probability density as:

$$P[\varphi, \tau] = \sum_0^\infty P_k[\varphi, \tau]. \tag{102}$$

So, the n-point correlation function in the kth order of perturbation theory will be:

$$\langle \varphi(\tau_1, x_1) \varphi(\tau_2, x_2) \cdots \varphi(\tau_n, x_n) \rangle_\eta = \lambda^k \int [d\varphi] \, \varphi(\tau_1, x_1) \varphi(\tau_2, x_2) \cdots \varphi(\tau_n, x_n) P_k[\varphi, \tau]. \tag{103}$$

The Fokker-Planck equation becomes:

$$\frac{\partial}{\partial \tau} P_k(\varphi, \tau) = \int d^d x \frac{\delta}{\delta \varphi(x)} \left(\frac{\delta}{\delta \varphi(x)} + \frac{\delta S_0}{\delta \varphi(x)} \right) P_k(\varphi, \tau) +$$
$$+ \int d^d x \frac{\delta}{\delta \varphi(x)} \frac{\delta S_I}{\delta \varphi(x)} P_{k-1}(\varphi, \tau), \tag{104}$$

with the initial conditions:

$$P_0[\varphi, 0] = \prod_y \delta(\varphi(y)), \tag{105}$$

$$P_k[\varphi, 0] = 0, \; k = 1, 2, \cdots \tag{106}$$

We may write a formal solution to the Fokker-Planck equation as:

$$P_k[\varphi, \tau] = \int [d\varphi'] \int_0^\tau d\tau' \, \mathcal{D}_0[\varphi, \varphi', \tau - \tau'] \int dx \, \frac{\delta}{\delta \varphi'(x)} \frac{\delta S_I}{\delta \varphi'(x)} P_{k-1}(\varphi', \tau'), \quad (107)$$

where the Green functional of the free Fokker-Planck equation satisfies:

$$\frac{\partial}{\partial \tau} \mathcal{D}_0[\varphi, \varphi', \tau - \tau'] = \prod_x \delta(\varphi(x) - \varphi'(x)) \, \delta(\tau - \tau') +$$

$$+ \int d^d x \frac{\delta}{\delta \varphi(x)} \left(\frac{\delta}{\delta \varphi(x)} + \frac{\delta S_0}{\delta \varphi(x)} \right) \mathcal{D}_0[\varphi, \varphi', \tau - \tau'], \quad (108)$$

with the boundary condition:

$$\mathcal{D}_0[\varphi, \varphi', \tau - \tau']|_{\tau = \tau'_+} = \prod_x \delta(\varphi(x) - \varphi'(x)). \quad (109)$$

For our case, the above equations should be modified in order to take into account the presence of the memory kernel. For instance, the free Fokker-Planck equation will be:

$$\frac{\partial}{\partial \tau} P_0[\varphi, \tau] = \int d^d x \int d^d y \frac{\delta}{\delta \varphi(x)} \left(\frac{\delta}{\delta \varphi(y)} M(x - y, \tau) + \frac{\delta}{\delta \varphi(y)} N[\varphi \, ; \tau, x] \right) P_0[\varphi, \tau], \quad (110)$$

where:

$$M(x - y, \tau) \equiv \int_0^\tau ds \, K_\Lambda(\tau - s) \, G(\tau - s, x - y), \quad (111)$$

and

$$N[\varphi \, ; \tau, x] \equiv \delta(x - y) \int_0^\tau ds \, K_\Lambda(\tau - s) S_0 \, |_{\varphi(x) = \varphi(s, x)} \, . \quad (112)$$

Now, we can easily see why we required earlier that β, defined in Eq.(73), should be real. Remembering the familiar transformation, $P[\varphi, \tau] = \psi[\varphi, \tau] \exp\left(-\frac{\hat{S}[\varphi]}{2}\right)$, with $\hat{S} \equiv M^{-1} N$ ($M = M(x - y, \tau)$, $N = N[\varphi \, ; \tau, x]$), we can recast Eq.(110) into the Schroedinger type equation:

$$\dot{\psi} = -2\mathcal{H}\psi, \quad (113)$$

where $\mathcal{H}$ is the Fokker-Planck Hamiltonian:

$$\mathcal{H} \equiv \frac{1}{2} \int d^d x \int d^d y \, Q(x) M(x - y, \tau) Q(y), \quad (114)$$

where:

$$Q(x) \equiv \hat{\Pi}(x) + \frac{1}{2} \left[\hat{\Pi}(x), \hat{S} \right], \quad (115)$$

20

and $\hat{\Pi}(x) \equiv -i\frac{\delta}{\delta\varphi(x)}$. We stress here that this derivation is purely formal. Now, let us study the function $M(x - y, \tau)$. In the Fourier space, it is given by:

$$M(k, \tau) = \frac{\Lambda^2}{2} \frac{1}{9\Lambda^4 + \beta^2} \Big\{ 8\Lambda^2 - $$
$$4 \exp\Big(-\frac{3\Lambda^2}{2}\tau\Big) \Big[2\Lambda^2 \cos\Big(\frac{\beta\tau}{2}\Big) + \Big(\frac{3\Lambda^4}{2\beta} - \frac{\beta}{2}\Big) \sin\Big(\frac{\beta\tau}{2}\Big) \Big] \Big\}, \qquad (116)$$

where we have used Eq.(71) and Eq.(74). So, we see that, in the limit $\tau \to \infty$, the M-function will be positive definite which, according to Eq.(114), guarantees that the Fokker-Planck Hamiltonian be positive definite as well. In other words, the real part of its eigenvalues will be positive. But what happens if β is a purely imaginary number? As for the Green function, we will have:

$$G(k, \tau) = \Big(\frac{\Lambda^2}{\beta} \sinh\Big(\frac{\beta\tau}{2}\Big) + \cosh\Big(\frac{\beta\tau}{2}\Big) \Big) \exp\Big(-\tau\frac{\Lambda^2}{2}\Big) \theta(\tau). \qquad (117)$$

So, from Eq.(116) and Eq.(117), it is easy to see that $M(k, \tau)$ may not be positive definite and, therefore, neither will be $\mathcal{H}$. Then, there is no guarantee that the real part of the eigenvalues of the latter will be positive. In other words, we cannot assure that, in the limit $\tau \to \infty$, the system will reach its ground state, i.e., that it converges to an equilibrium.

Now, let us study the solution to Eq.(110). It will be of the form:

$$P_0[\varphi, \tau] = \int [d\eta] \prod_y \delta(\varphi(y) - \varphi(\tau, y))$$
$$\exp\Big(-\frac{1}{4} \int d^dx \int d\tau'' \int d\tau' \, \eta(\tau'', x) \, K_\Lambda^{-1}(\tau'' - \tau') \, \eta(\tau', x)\Big), \qquad (118)$$

where $\varphi(\tau, y)$ satisfies Eq.(62). We can rewrite this last equation in the form:

$$P_0[\varphi, \tau] = \int [d\eta] \int [d\xi] \exp\Big(i \int d^dx \, \xi\Big(\varphi(x) - \varphi(\tau, x)\Big)\Big)$$
$$\exp\Big(-\frac{1}{4} \int d^dx \int d\tau'' \int d\tau' \, \eta(\tau'', x) \, K_\Lambda^{-1}(\tau'' - \tau') \, \eta(\tau', x)\Big). \qquad (119)$$

Using Eq.(62) and doing the η, ξ functional integrations, we obtain:

$$P_0[\varphi, \tau] = N_0^{-1} \exp\Big(-\frac{1}{2} \int d^dx \int d^dx' \varphi(x) \, D^{-1}(\tau, x; \tau, x')\varphi(x')\Big), \qquad (120)$$

21

where the D-function is just the free two-point correlation function and N_0^{-1} is a normalization factor. In momentum space, we get, for $\tau \to \infty$:

$$P_0[\varphi, \tau] = N_0^{-1} \exp\left(-\frac{1}{4} \int d^d k \, \varphi(k)(k^2 + m_0^2)\varphi(-k)\right). \tag{121}$$

As we can see, this result resembles the usual equilibrium result, up to a constant, in contrast with the result obtained by Breit et al [22]. Now, proceeding with similar steps (see reference [8]), it is possible to show that the Green functional for the free Fokker-Planck equation is given by (Δ_0^{-1} is a normalization factor):

$$\mathcal{D}_0[\varphi, \varphi', \tau] = \Delta_0^{-1} \exp\left(-\frac{1}{2} \int d^d k \, \varphi_i(k) D_{ij}(\tau)\varphi_j(-k)\right), \tag{122}$$

where:

$$\varphi_i \equiv \left(\, \varphi(k) \quad \varphi'(k)\,\right), \tag{123}$$

and the elements of the matrix D_{ij} are given by:

$$D_{ij}(\tau) \equiv \begin{pmatrix} D(k; \tau, \tau) & -D(k; \tau, \tau)\, G(k, \tau) \\ -D(k; \tau, \tau)\, G(k, \tau) & D(k; \tau, \tau)\, G(k, \tau)\, G(k, \tau) \end{pmatrix}. \tag{124}$$

It is easy to verify that, in the limit $\tau \to \infty$, P_0 and $\mathcal{D}_0$ will satisfy, up to constants, similar relations as obtained by Floratos and Iliopoulos [8]. This behavior can be understood if we notice the similar structure between the Green function of the usual Parisi-Wu scheme and our non-Markovian approach. The same is true for both free two-point correlation functions. However, for massless scalar theories, those estimations in [8] decay as inverse power of τ. In our approach, even in this massless situation, an exponential behavior is found. Therefore, in the limit $\tau \to \infty$, we get an improved convergence.

From all of these results we expect to obtain probability functions in the perturbation theory that are very close to those usual equilibrium probability densities. Therefore the theory will not be regularized. As remarked on [16], nothing is lost by using a non-Markovian description in place of a Markov description, as long as it is Gaussian.

7 Conclusions

It is well known that the Langevin equation given by Eq.(32) is only one particular choice in a large class of relaxation equations. The aim of this paper was to investigate if the Parisi-Wu quantization method can be extended assuming a Langevin equation with a memory kernel with the modified Einstein relations. Therefore, in this paper we discussed the stochastic quantization for massive self-interacting scalar field, first assuming a Markovian Langevin equation. Then, we studied the $(\lambda\varphi^4)$ theory at the one-loop level introducing a non-Markovian Langevin equation and examine the interacting field theory that appears in the asymptotic limit of the non-Markovian process.

To make sure that the first modification can be used, one must first check that the system evolves to the equilibrium in the asymptotic limit. Second we have to show that converges to the correct equilibrium distribution. We proved that although the system evolves to equilibrium, we obtain a non-regularized theory. In contrast with dimensional regularization [32] [33] [34] [35] and also other regularization procedures, a remarkable property of the stochastic regularization is that it preserves all the symmetries of the original theory. With our results, it is easy to see that the system described by this non-Markovian Langevin equation converges, but the virtues of the stochastic regularization that we hoped to appear in this framework are lost.

A natural continuation of this paper is to discuss the stochastic quantization of bosonic and femionic fields in general Riemannian spaces. A different application of the stochastic quantization is to discuss interacting field theory in the presence of macroscopic structures. It is well known that performing the weak-coupling perturbative expansion, to renormalize the interacting field theory, we have to introduce not only the usual bulk counterterms, but also surface counterterms. This can be done at the one-loop level at zero and finite temperature [36] [37] [38]. To extend the calculation to high-order loops will appear overlapping divergences. A natural question is how to implement the stochastic quantization in systems with macroscopic structures. This subject is under investigation by the authors.

8 Acknowledgements

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